The Operational Calculus

CONVOLUTION ALGEBRA

With the operational notation any real-valued, defined and continued function $f(t)$ of the variable $t \geq 0$ is represented by the symbol $\{f\}$. The operations of addition and of multiplication are defined by

$$\{f\} + \{g\} = \{f + g\},$$

$$\{f\}\ \{g\} = \left\{\int_0^t f(\tau) g(t-\tau)\, d\tau\right\}.$$

It is easy to verify that both operations are commutative and associative, that the multiplication is distributive with respect to addition, that both operations are closed, i. e., the sum or the product of two functions defined and continuous for $t \geq 0$ is a function of the same class. Finally for any function $\{f\}$ there exists an additive identity, i. e.,

$$\{f\} + \{0\} = \{f\}$$

and an additive inverse, i. e.,

$$\{f\} + \{-f\} = \{0\}.$$

It is useful to define the *power* of a function with the usual notation

$$\{f\}^2 = \{f\} \cdot \{f\},$$

$$\{f\}^{n+1} = \{f\}^n \cdot \{f\}, \quad n = 2, 3, \ldots$$

Using the above definition we can compute

$$\{1\}^2 = \left\{\int_0^t 1 \cdot 1 \cdot d\tau\right\} = \{t\},$$

$$\{1\}^3 = \left\{\int_0^t 1 \cdot (t-\tau) \cdot d\tau\right\} = \left\{t^2/2\right\},$$

and in general

$$\{1\}^n = \left\{\frac{t^{n-1}}{(n-1)!}\right\}$$

for any positive integer n.

QUOTIENT OF FUNCTIONS

We define the *quotient* of two functions,

$$\frac{\{f\}}{\{g\}} = \{h\},$$

with $\{g\} \neq \{0\}$, if a function $\{h\}$ exists such that

$$\{f\} = \{g\} \cdot \{h\}.$$

It can be proved that if two functions have a quotient, it is unique.

To make the operation of quotient of two functions always possible, with the only restriction that the divisor be different from $\{0\}$, we define the *operator* by the following properties:

$$\frac{\{f\}}{\{g\}} = \frac{\{\varphi\}}{\{\psi\}} \text{ if and only if } \{f\} \cdot \{\psi\} = \{g\} \cdot \{\varphi\}, \tag{1}$$

$$\frac{\{f\}}{\{g\}} + \frac{\{\varphi\}}{\{\psi\}} = \frac{\{f\} \cdot \{\psi\} + \{g\} \cdot \{\varphi\}}{\{g\} \cdot \{\psi\}},$$

$$\frac{\{f\}}{\{g\}} \cdot \frac{\{\varphi\}}{\{\psi\}} = \frac{\{f\} \cdot \{\varphi\}}{\{g\} \cdot \{\psi\}}.$$

Definition (1) shows that the same operator can be written in different forms; for instance $\{2t\}/\{t\}$ and $\{t^2\}/\{t\}$ are the same operator because $\{2t\} \cdot \{t\} = \{t\} \cdot \{t^2\}$.

From what we have seen so far, all functions of the form $\{f\}$ have the same formal properties of the integers, and all operators of the form $\{f\}/\{g\}$, with $\{g\} \neq \{0\}$, have the same formal properties of the rational numbers.

NUMERICAL OPERATOR

For any constant a and both $\{f\}$ and $\{g\}$ different from $\{0\}$, we can write

$$\frac{\{a \cdot f\}}{\{f\}} = \frac{\{a \cdot g\}}{\{g\}},$$

as a consequence of definition (1). Therefore this operator depends only on the constant a; we shall write

$$\frac{\{a \cdot f\}}{\{f\}} = a$$

by definition and call it the *numerical operator*.

It can be proved that the numerical operator has all formal properties of the real numbers.

Observe that, using the properties listed so far,

$$0 = \frac{\{0\}}{\{f\}} = \frac{\{0\} \cdot \{f\}}{\{f\}} = \{0\};$$

this shows that the numerical operator 0 is also equal to the constant function $\{0\}$; this is the only numerical operator that can be written with or without brackets.

DIFFERENTIAL OPERATOR

If $\{f\}/\{g\}$ is an operator with $\{f\} \neq 0$, its *inverse* is the operator $\{g\}/\{f\}$. Obviously,

$$\frac{\{f\}}{\{g\}} \cdot \frac{\{g\}}{\{f\}} = 1 .$$

Because of the identity

$$\{1\} \cdot \{f\} = \left\{ \int_0^t f(\tau)\, d\tau \right\},$$

we call $\{1\}$ the *integral operator*.

We define the operator s as the inverse of $\{1\}$,

$$s \cdot \{1\} = 1$$

and call it the *differential operator*; obviously,

$$\left\{ \int_0^t f(t)\, dt \right\} = \frac{\{f\}}{s} . \tag{2}$$

For any function $f(t)$ having a derivative $f'(t)$ we can write

$$\int_0^t f'(\tau)\, d\tau = f(t) - f(0) ,$$

and in operational form

$$\{1\} \cdot \{f'\} = \{f\} - \{f(0)\} ,$$

and by dividing both sides by $\{1\}$,

$$\{f'\} = s\{f\} - f(0) . \tag{3}$$

By applying this formula several times we can write

$$\{f^{(n)}\} = s^n \{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) .$$

FUNCTIONAL CORRELATES

From the definition of the differential operator,

$$\{1\} = \frac{1}{s} ;$$

multiplying both sides by $\{1\}$ we get

$$\{t\} = \frac{1}{s^2} ;$$

multiplying again,

$$\{t^2\} = \frac{2}{s^3}$$

and by induction,

$$\{t^n\} = \frac{n!}{s^{n+1}}.$$

From the identities

$$\frac{de^{\alpha t}}{dt} = \alpha \cdot e^{\alpha t},\ e^0 = 1$$

we get

$$\{\alpha \cdot e^{\alpha t}\} = s \cdot \{e^{\alpha t}\} - 1,$$

thence

$$\{e^{\alpha t}\} = \frac{1}{s - \alpha}.$$

From the identities

$$\frac{d \sin \beta t}{dt} = \beta \cos \beta t,\ \frac{d \cos \beta t}{dt} = -\beta \sin \beta t,$$
$$\sin 0 = 0,\ \cos 0 = 1,$$

we get

$$\{\sin \beta t\} = \frac{\beta}{s^2 + \beta^2},\ \{\cos \beta t\} = \frac{s}{s^2 + \beta^2}.$$

ALGEBRAIC DERIVATIVE OF AN OPERATOR

The operation D on a function is defined by

$$\mathrm{D}\{f\} = \{-t \cdot f\};$$

D is not an operator because, in general,

$$\frac{\{-t \cdot f\}}{\{f\}} \neq \frac{\{-t \cdot g\}}{\{g\}}.$$

The operation D has the properties

$$\mathrm{D}\big(\{f\}+\{g\}\big)=\mathrm{D}\{f\}+\mathrm{D}\{g\}$$
$$\mathrm{D}\big(\{f\}\cdot\{g\}\big)=\mathrm{D}\{f\}\cdot\{g\}+\{f\}\cdot\mathrm{D}\{g\}$$
$$\text{If } \{f\}=\frac{\{g\}}{\{h\}} \text{ then } \mathrm{D}\{f\}=\frac{\mathrm{D}\{g\}\cdot\{h\}-\{g\}\cdot\mathrm{D}\{h\}}{\{h\}^2}$$

that are easy to prove by substitution.

The operation D on an operator is defined by

$$\text{If } p=\frac{\{f\}}{\{g\}} \text{ then } \mathrm{D}p=\frac{\mathrm{D}\{f\}\cdot\{g\}-\{f\}\cdot\mathrm{D}\{g\}}{\{g\}^2};$$

it is easy to prove that

$$\mathrm{D}(p+q)=\mathrm{D}p+\mathrm{D}q,$$
$$\mathrm{D}(p\cdot q)=\mathrm{D}p\cdot q+p\cdot\mathrm{D}q,$$
$$\mathrm{D}\left(\frac{p}{q}\right)=\frac{\mathrm{D}p\cdot q-p\cdot\mathrm{D}q}{q^2}.$$

As a consequence, for a numerical operator a we have

$$\mathrm{D}a=\mathrm{D}\left(\frac{\{a\}}{\{1\}}\right)=\frac{\mathrm{D}\{a\}\cdot\{1\}-\{a\}\cdot\mathrm{D}\{1\}}{\{1\}^2}=0\,.$$

For the differential operator we have

$$\mathrm{D}s=\mathrm{D}\left(\frac{1}{\{1\}}\right)=\frac{\mathrm{D}1\cdot\{1\}-1\cdot\mathrm{D}\{1\}}{\{1\}^2}=1\,.$$

By induction we can prove that

$$\mathrm{D}\{s\}^n=n\cdot s^{n-1}\,.$$

All these properties of the operation D are formally identical to the derivative with respect to s, therefore, if R(s) is a rational expression in s, we can write

$$\mathrm{DR}(s)=\frac{d\mathrm{R}(s)}{ds}\,. \tag{4}$$

INITIAL VALUE THEOREM

As shown by Mikusinski,([1])

$$\int_0^\infty e^{-\lambda s} f(\lambda)\,d\lambda=\{f\}\,,$$

thence, using (3),

$$\int_0^\infty e^{-\lambda s} f'(\lambda)\,d\lambda = s\{f\} - f(0) \tag{5}$$

and

$$\lim_{s\to\infty}\int_0^\infty e^{-\lambda s} f'(\lambda)\,d\lambda = \lim_{s\to\infty}\left[s\{f\} - f(0)\right].$$

Since λ and s are independent, we can change the order of the limit process on the left-hand side, therefore

$$f(0) = \lim_{s\to\infty} s\{f\}.$$

FINAL VALUE THEOREM

From (5) we can write

$$\lim_{s\to 0}\int_0^\infty e^{-\lambda s} f'(\lambda)\,d\lambda = \lim_{s\to 0}\left[s\{f\} - f(0)\right].$$

Since λ and s are independent, we can change the order of the limit process on the left-hand side, therefore

$$\int_0^\infty f'(\lambda)\,d\lambda = \lim_{s\to 0}\left[s\{f\} - f(0)\right];$$

but

$$\int_0^\infty f'(\lambda)\,d\lambda = \lim_{t\to\infty} f(t) - \lim_{t\to 0} f(t),$$

thence

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\{f\}. \tag{6}$$

Reference

[1] J. Mikusinski. Operational Calculus, Pergamon Press, London, 1959